EXTENSIONS OF DESCARTES' RULE OF SIGNS CONNECTED WITH A PROBLEM SUGGESTED BY LAGUERRE*

BY

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1. A Problem Proposed by Laguerre

As one of the first applications of his proof of Descartes' Rule of Signs in a form admitting immediate extension to power series, Laguerre† considered certain properties of an infinite rectangular array formed from the coefficients of the developments in descending powers of x of the expressions

$$\frac{f(x)}{(x-1)^i} \qquad (i = 1, 2, \cdots),$$

where $f(x)$ is a given polynomial of degree n, and $f(1) \neq 0$. Thus if we write, with a notation differing slightly from Laguerre's,

$$\frac{f(x)}{(x-1)^i} = \sum_{j=0}^{\infty} a_{ij}\, x^{n-i-j},$$

and form the array

$$\begin{matrix} a_{10} & a_{11} & a_{12} & \cdot & \cdot & \cdot \\ a_{20} & a_{21} & a_{22} & \cdot & \cdot & \cdot \\ a_{30} & a_{31} & a_{32} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}, \tag{1}$$

Laguerre has proved that a superior limit for the number of roots of $f(x)$ in the interval $[1, +\infty]$ is given by the number of variations of sign in the sequence consisting of any horizontal line of (1). If the latter number is greater than the former, the difference is an even number. Laguerre also proves that another type of sequence having the same property is obtained by proceeding along any horizontal line until we reach a term a_{ij} for which $i + j$ is at our choice, subject to the inequality $i + j \geqq n + 1$, and then taking a path diagonally upward to the right. These results we generalize somewhat in § 3.

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† *Oeuvres*, vol. 1, pp. 1–25.

Laguerre notes that the number v of variations of sign in a row of (1) either decreases or remains stationary as we go down in the array, and hence must have a minimum greater than or equal to the number m of roots of $f(x)$ in $[1, +\infty]$. He remarks on the importance of determining this minimum, but considers the problem one of great difficulty.* This problem remained unsolved until very recently, when it was proved by Fekete and Pólya† that if we go sufficiently far down in the array (1), we shall have $v = m$.

The proofs of Fekete and Pólya involve rather intricate systems of inequalities, though they have the advantage of extending the results to cases where $f(x)$ is not a polynomial. It is the purpose of the present paper to present a simpler proof modeled on Laguerre's treatment of a similar problem, where he shows that the number of variations of sign presented by the coefficients in the development of $e^{zx} f(x)$ according to ascending powers of x is equal to the number of positive roots of $f(x)$ provided z is sufficiently large.‡ This method assumes that $f(x)$ is a polynomial, and additional considerations would be necessary to extend the proof to other cases.

In addition to this solution of Laguerre's problem, other results of a similar nature connected with arrays analogous to (1) are obtained in the present paper.

2. Definitions for Laguerre Sequences

We shall here slightly generalize the array (1) by considering the expansions

$$\frac{f(x)}{(x-z)^i} = \sum_{j=0}^{\infty} A_{ij}\, x^{n-i-j} \qquad (i = 0, 1, 2, \cdots), \tag{2}$$

where $f(x)$ is the polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \qquad (a_0 \neq 0), \tag{3}$$

and z is a positive number not a root§ of $f(x)$. The array to be studied is then

$$\begin{matrix} A_{00} & A_{01} & A_{02} & \cdot & \cdot & \cdot \\ A_{10} & A_{11} & A_{12} & \cdot & \cdot & \cdot \\ A_{20} & A_{21} & A_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \, . \tag{4}$$

We also generalize the sequences mentioned in § 1, and define as a *Laguerre*

* Loc. cit., p. 14.

† *Ueber ein Problem von Laguerre,* Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), p. 89. The problem is here stated in slightly different form.

‡ *Oeuvres,* vol. 1, pp. 22–25.

§ The following argument can easily be modified so as to include the case $f(z) = 0$, but the extra complications in statement make it hardly worth while here.

Sequence any sequence composed of the elements which lie on a path beginning anywhere in the first column below the first row of (4) and proceeding at each move either horizontally or diagonally upwards to the right until it reaches some row below the first, along which it proceeds indefinitely to the right; i. e., a sequence

$$A_{r_0 0}, \ A_{r_1 1}, \ A_{r_2 2}, \ \cdots, \tag{5}$$

where, for any i, r_{i+1} may either coincide with r_i or be less by unity than r_i, either case being possible at each step, but all the r's are greater than zero. It will be seen that this definition includes as special cases the two kinds of sequences studied by Laguerre. We shall show that every Laguerre Sequence has properties analogous to those mentioned for these two in § 1.

One more designation we shall find useful. A Laguerre Sequence (5) will be said to *enclose* a second Laguerre Sequence

$$A_{s_0 0}, \ A_{s_1 1}, \ A_{s_2 2}, \ \cdots,$$

if for every i we have $r_i \geqq s_i$; i. e., if each member of the first sequence either coincides with or lies below that member of the second sequence which is in the same column of (4).

3. Formulas for the Elements of the Array (4)

The first $n + 1$ elements of the first row of (4) are evidently the coefficients of $f(x)$, and the remaining elements of this row are zeroes; i. e., we have

$$\begin{aligned} A_{0j} &= a_j \qquad (j = 0, 1, \cdots, n), \\ &= 0 \qquad (j > n). \end{aligned} \tag{6}$$

Each of the following rows is obtained from its predecessor by the division algorithm. We thus obtain the formulas

$$\begin{aligned} A_{i0} &= A_{00} \qquad (i = 1, 2, \cdots), \\ A_{ij} &= A_{i-1\,j} + zA_{i\,j-1} \qquad (i > 0, j > 0). \end{aligned} \tag{7}$$

By repeated use of (7), or more easily by direct multiplication of (3) by the development of $(x - z)^{-i}$ in descending powers of x, we find that for every $j > 0$,

$$\begin{aligned} A_{ij} = A_{0j} + A_{0\,j-1}\, iz + A_{0\,j-2} \frac{i(i+1)}{2} z^2 + \cdots \\ + A_{00} \frac{i(i+1)\cdots(i+j-1)\, z^j}{j!}. \end{aligned} \tag{8}$$

As the terms of (4) on or beyond what Laguerre styles the *principal diagonal* (the diagonal passing through the elements for which $i + j = n + 1$) play a special rôle in what follows, we proceed, for these terms, to throw (8), with the help of (6), into the form

$$A_{ij} = \frac{(i+j-n-1)!}{(i-1)!j!} F(z, i, j) z^{j-n} \quad (i+j>n,\ i>0), \tag{9}$$

where

$$\begin{aligned} F(z, i, j) = {} & a_0(i+j-1)(i+j-2) \cdots (i+j-n) z^n \\ & + a_1 j(i+j-2) \cdots (i+j-n) z^{n-1} + \cdots \\ & + a_k j(j-1) \cdots (j-k+1)(i+j-k-1) \cdots (i+j-n) z^{n-k} \\ & + \cdots + a_n j(j-1) \cdots (j-n+1). \end{aligned} \tag{10}$$

Two other expressions for the elements of (4) are of interest. The first is given by the formula

$$A_{ij} = z^{j-n} S_j^{(i-1)} \qquad (i>0,\ j \geqq 0), \tag{11}$$

where $S_j^{(i-1)}$ denotes the numerator of the $(j+1)$th Cesàro mean* of order $i-1$ for the series

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n + 0 + 0 + \cdots.$$

The other formula, which applies to terms on or beyond the principal diagonal, is

$$A_{ij} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dz^{(i-1)}} (z^{i+j-n-1} f(z)) \quad (i>0,\ i+j>n). \tag{12}$$

This includes the well-known result that the elements of the principal diagonal verify the relation

$$A_{i\,n+1-i} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dz^{(i-1)}} f(z) \qquad (0<i<n+2).$$

From this latter formula the truth of (12) follows at once if we note that A_{ij} is on the principal diagonal of the array for the polynomial $x^{i+j-n-1} f(x)$.

4. The Number of Variations of Sign in Laguerre Sequences

From formulas (9) and (10) we derive the property expressed in the following theorem.

THEOREM 1. *For every $r > 0$ an integer $s > n$ can be determined such that every element of the array* (4) *situated in a row below the first and above the rth, and in a column beyond the sth, is of the same sign as $f(z)$.*

This follows immediately from the fact that $F(z, i, j)$, as given by (10), is a polynomial of degree n in j for which the coefficient of j^n is $f(z)$; and

* See Bromwich, *Theory of Infinite Series*, p. 311. Cf. also Fekete and Pólya, loc. cit., p. 101.

by hypothesis, $f(z) \neq 0$. Hence for each $i > 0$ there is an integer s_i such that A_{ij} is of the same sign as $f(z)$ for all $j \geqq s_i$. In the above theorem s can be taken as the greatest of the numbers $s_1, s_2, \cdots, s_{r-2}$.

From Theorem 1 it follows that every Laguerre Sequence terminates in an infinite number of terms of the same sign as $f(z)$, and consequently has only a finite number of variations of sign.

THEOREM 2. *If L_1 and L_2 are two Laguerre Sequences chosen arbitrarily from the array* (4) *subject to the condition that L_2 encloses L_1, and if V_1, V_2 are the numbers of variations of sign presented by L_1, L_2 respectively, then $V_1 - V_2 = 0$ or an even positive integer.*

To prove this theorem we first note that by formula (7) the sequence

$$A_{i\,j-1},\ A_{i-1\,j},\ A_{i-1\,j+1}$$

presents either the same number of variations of sign, or else two more than

$$A_{i\,j-1},\ A_{i\,j},\ A_{i-1\,j+1},$$

provided $A_{i-1\,j+1} \neq 0$. Hence if L_2 and L_1 are the same except that one term A_{ij} of the former is replaced by $A_{i-1\,j}$ in the latter, the conclusion of Theorem 2 must be true.* This will be the case even when $A_{i-1\,j+1} = 0$, for the number of variations of sign in both L_1 and L_2 would be unchanged if $A_{i-1\,j+1}$ were given the sign of the first succeeding term that does not vanish; by Theorem 1 such terms always exist.

If L_1 and L_2 are now taken in more general fashion, but so that they coincide after k terms, where k is any positive integer, the former sequence may be transformed into the latter by a finite number of substitutions of terms A_{ij} for terms $A_{i-1\,j}$, each such substitution leaving the number of variations of sign unchanged, or decreasing it by two.

Finally, if L_1 and L_2 do not coincide after k terms, no matter how large k may be, we use the conclusion of Theorem 1 to replace L_2 by a sequence L_3 which coincides with L_2 until we have entered that portion of the array where all terms are of the same sign, and then proceeds diagonally upward until a term of L_1 is reached, after which it coincides with L_1. Since the conclusion of the previous paragraph holds for L_1 and L_3, and L_2 and L_3 have the same number of variations of sign, we have now established Theorem 2 for all cases.

THEOREM 3. *Let L be an arbitrarily chosen Laguerre Sequence presenting V variations of sign, and let M be the number of real roots of $f(x)$ in the interval $[z, +\infty]$. Then $V - M = 0$ or an even integer.*

Laguerre has proved† this property for every sequence composed of the

* The case where $j = 0$ is easily disposed of, since $A_{i-1\,0} = A_{i0}$.

† Loc. cit., p. 14.

terms of a row of (4). An arbitrary Laguerre Sequence is enclosed by sequences of this type, hence Theorem 3 follows as a consequence of Theorem 2.

5. Proofs of the Existence of Laguerre Sequences for which $V = M$

We now present a proof, based on Laguerre's methods in a similar case (see § 1), that there are sequences for which V, the number of variations of sign, is exactly equal to M, the number of roots of $f(x)$ in $[z, +\infty]$.

Theorem 4. *There exists a positive integer $r > n$, such that for the sequence composed of the $(r+1)$th horizontal row of* (4) *we have $V = M$.*

If we refer to the notation of formulas (9) and (10) we see that V_i, the number of variations of sign in the $(i+1)$th row of (4), is equal to the number of variations of sign in the sequence composed of the values of $F(z, i, j)$ for $j = 0, 1, 2, \cdots$. Make the substitutions

$$\frac{j}{i} = x, \qquad \frac{1}{i} = \omega .$$

Then we have

$$\frac{1}{i^n} F(z, i, j) = \phi(z, x, \omega), \tag{13}$$

where

$$\begin{aligned}\phi(z, x, \omega) &= a_0(1 + x - \omega)(1 + x - 2\omega) \cdots (1 + x - n\omega) z^n + \cdots \\ &+ a_k x(x-\omega)\cdots(x-(k-1)\omega)(1+x-(k+1)\omega)\cdots(1+x-n\omega)z^{n-k} + \cdots \\ &+ a_n x(x-\omega)\cdots(x-(n-1)\omega).\end{aligned} \tag{14}$$

The number V_i is equal to the number of variations of sign in the sequence formed by the values of $\phi(z, x, \omega)$ for $x = 0, \omega, 2\omega, \cdots$, so that $V_i \leqq M_\omega$, where M_ω denotes the number of positive roots of $\phi(z, x, \omega)$ regarded as a polynomial in x.

We now observe that

$$\phi(z, x, 0) = x^n f\left(\frac{1+x}{x} z\right),$$

so that the number M_0 of positive roots of $\phi(z, x, 0)$ regarded as a polynomial in x is equal to M, the number of roots of $f(x)$ in $[z, +\infty]$. We will prove that if ω_1 is a sufficiently small positive number, we must have $M_\omega \leqq M_0$ for all positive values of ω less than ω_1, so that $M_\omega \leqq M$. Since we have seen above that $V_i \leqq M_\omega$, it follows that $V_i \leqq M$ if i is sufficiently large. But from Theorem 3 we have $V_i \geqq M$, and the proof of our theorem is thus completed when we have established the point above mentioned.

We now show that the number ω_1 mentioned in the preceding paragraph may be chosen as follows: Designate by $\Theta(\omega)$ the discriminant of $\phi(z, x, \omega)$ regarded as a polynomial in x; then ω_1 may be taken as a positive number not greater than $1/n$, and not greater than the smallest positive root of $\Theta(\omega)$. In the notation of the preceding paragraphs, it remains to prove that $M_\omega \leqq M_0$ for all positive $\omega < \omega_1$. Let ω_0 be such a value of ω, and let ω increase steadily from $\omega = 0$ to $\omega = \omega_0$. The n roots of $\phi(z, x, \omega)$ regarded as a polynomial in x are continuous functions of ω. A glance at (14) shows that $x = 0$ can not be a root of $\phi(z, x, \omega)$ for $\omega < \omega_1 \leqq 1/n$. From this it follows that a root which is negative for $\omega = 0$ can not become positive when ω has increased from 0 to ω_0. A root imaginary for $\omega = 0$ cannot become real for $\omega = \omega_0$, since this would imply the existence of a value of ω between 0 and ω_0 for which the root coincides with its conjugate; but this is impossible because the discriminant $\Theta(\omega)$ does not vanish in $[0, \omega_0]$. Thus the only roots of $\phi(z, x, \omega)$ that can be positive real numbers for $\omega = \omega_0$ are those which were real and positive for $\omega = 0$. In other words, we must have $M_{\omega_0} \leqq M_0$. This completes the proof of Theorem 4; we have in fact shown that r may be chosen as any integer $> 1/\omega_1$.

THEOREM 5. *There exists a positive integer $s > n$ such that the "diagonal sequence" beginning with the term A_{s0} and composed of terms on a line parallel to the principal diagonal of* (4), *followed by terms of the second row, has the property that $V = M$.*

The truth of this follows at once from Theorems 1 and 4. Let r be a positive integer chosen in accordance with Theorem 4. We may replace the sequence L_1 of terms of the $(r + 1)$th row of (4) by a sequence L_2 which coincides with L_1 until that part of the array is reached where, by Theorem 1, all terms are of the same sign, after which L_2 proceeds diagonally upward to the second row and thereafter remains in that row. For L_2 we also have $V = M$. If L_2 is now replaced by the diagonal sequence L_3 composed of *all* the terms, up to the second row, in the diagonal along part of which L_2 proceeded from the $(r + 1)$th row to the second row, followed by terms of the latter row, we see that L_3 encloses L_2, and hence cannot have more variations of sign than L_2. But by Theorem 3 we have $V \geqq M$ for every Laguerre Sequence; hence for L_3 we have $V = M$, and L_3 satisfies the requirements of Theorem 5.

THEOREM 6. *There exists a Laguerre Sequence L for which $V = M$, and such that all sequences which enclose L have the same property, while all other Laguerre Sequences have $V > M$.*

The sequence L here indicated may be obtained by taking a diagonal sequence which satisfies Theorem 5 and performing on it successive substitutions of the type considered in the proof of Theorem 2, i. e., substitutions of a term $A_{i-1\,j}$ for A_{ij} which are such that the new sequence is a Laguerre Sequence. Only such substitutions are, however, to be made as produce at each step a new sequence with V unchanged. When the last possible substitution in a succession of this kind has been made the result will be the sequence

L of Theorem 6.* To prove this we note first that all enclosing sequences can have no more variations of sign than L, by Theorem 2, and no less than M variations of sign, by Theorem 3, so that $V = M$ for every sequence enclosing L. If a sequence (other than L itself) is enclosed by L it is obtained from L by substitutions of terms of the type above considered, and therefore must have more variations of sign than L. The only case remaining is that of a sequence $\bar{L}$ which neither encloses L nor is enclosed by L. The sequences L and $\bar{L}$ must then have one or more terms common. Let these terms be designated, in the order in which they occur in L, by $A_{i_1j_1}, A_{i_2j_2}, \cdots$. The sequence of terms of L from its first term to the term $A_{i_1j_1}$ inclusive we will style the *segment* L_0, the sequence from $A_{i_1j_1}$ to $A_{i_2j_2}$ inclusive the *segment* L_1, etc. If there is a last common term $A_{i_kj_k}$, the *segment* L_k will begin with this term and include all remaining terms of L. We similarly define segments $\bar{L}_0, \bar{L}_1, \cdots$, of the sequence $\bar{L}$. We shall say that L_p encloses $\bar{L}_p$ if for each corresponding pair of terms in the same column of (4) the term of L_p is in the same row or a lower row than that where the term of $\bar{L}_p$ is situated; similarly if $\bar{L}_p$ encloses L_p. Since $\bar{L}$ does not enclose L nor coincide throughout with it, there must be at least one segment $\bar{L}_p$ enclosed by the corresponding L_p and not coincident with the latter. For such a segment $\bar{L}_p$ the number of variations of sign presented by its terms must be greater than that presented by L_p, otherwise the sequence obtained by replacing in L the segment L_p by the segment $\bar{L}_p$ would be a Laguerre Sequence enclosed by L and having no greater number of variations of sign than L; but this we have seen to be impossible. On the other hand no segment $\bar{L}_q$ which encloses L_q can present less variations of sign than L_q, otherwise the substitution in L of $\bar{L}_q$ for L_q would produce a Laguerre Sequence having less variations of sign than L, which is again impossible. Since all segments of $\bar{L}$ must belong to one of the two classes here considered, and the number of variations of sign in a sequence is the sum of the numbers of variations in a complete set of segments, it follows from the above that $\bar{L}$ must have more variations of sign than L. The proof of Theorem 6 is thus completed.

Let us note here that in all the theorems of this paper we may replace the array (4) by one in which the elements below the first row are the Cesàro means $S_j^{(i-1)}$ (see formula (11)). On account of the importance of Cesàro means in certain questions in other fields, it is of interest to find them playing a rôle in the theory of equations.

6. Laguerre Sequences Analogous to Sequences of Sturm's Functions

The terms A_{ij} of the array (4) are polynomials in z. The question at once presents itself as to how the number of variations of sign in a Laguerre

* Our construction process leads, therefore, to a *unique* result.

Sequence varies with z. In particular, since the number of roots in $[z, +\infty]$ remains stationary or decreases as z increases, it might be hoped that the number of variations of sign in a Laguerre Sequence would always tend to decrease as z increases. A simple example will show that this is not true. Let us take $f(x) = x^2 - 3x + 2$; the second row of (4) is then

$$1 \quad z-3 \quad f(z) \quad zf(z) \quad \cdot \quad \cdot \quad \cdot.$$

For $z = 3/2$ this Laguerre Sequence has one variation of sign, but for $z = 5/2$ there are two. Thus there are even sequences of the type mentioned in Theorem 6 for which our supposition is untrue. We can, however, easily establish the truth of the following statement.

THEOREM 7. *For each positive $z = z_0$ not a root of $f(z)$ there exists a positive integer r such that every Laguerre Sequence beginning in a row of* (4) *below the rth has for every $z \geqq z_0$, subject to the restriction $f(z) \neq 0$,* exactly as many variations of sign as $f(x)$ has roots in $[z, +\infty]$.*

To prove this we first note that if $\phi(x)$ is any polynomial of degree m whose *Fourier Sequence,*

$$\phi(x), \qquad \frac{d}{dx}\phi(x), \qquad \cdots, \qquad \frac{d^m}{dx^m}\phi(x)$$

has the number V_z of variations of sign for $x = z$, and if a is such a number (not a root of $\phi(x)$) that $V_a = M_a$, where M_a is the number of roots of $\phi(x)$ in $[a, +\infty]$, then for all $b > a$ such that $\phi(b) \neq 0$ we shall have $V_b = M_b$. This follows from the well-known theorem of Fourier which is expressed by the formula

$$V_a - V_b \geqq M_a - M_b.$$

Since $V_a = M_a$, we must have $V_b \leqq M_b$; but as a corollary of Fourier's theorem $V_b \geqq M_b$; hence $V_b = M_b$.

By formula (12) a diagonal sequence having for $z = z_0$ the property expressed in Theorem 5 is composed of the terms, in reversed order and each multiplied by a positive constant, of the Fourier Sequence for $z^k f(z)$ where k is a positive integer, followed by terms of the same sign as $f(z)$. Hence from the above remark on Fourier Sequences it follows that a diagonal sequence verifying Theorem 5 for $z = z_0$ satisfies the requirements of Theorem 7. From Theorems 2 and 3 the same must be true for every Laguerre Sequence enclosing such a diagonal sequence, i. e., for every sequence of the type mentioned in Theorem 7.

This argument also establishes the truth of the following statement.

THEOREM 8. *Each of the sequences indicated in Theorem 7 has the property*

* This restriction, imposed by the form of our previous results, may easily be removed. In that case Theorem 7 will still be true if $[z, +\infty]$ is defined as not including z.

for the interval $[z_0, +\infty]$ *possessed by sequences of Sturm's functions, that for every subinterval* $[a, b]$ *where* $f(a) \neq 0, f(b) \neq 0$,* *the number of variations of sign for* $z = a$ *diminished by the number for* $z = b$ *is equal to the number of roots of* $f(x)$ *in* $[a, b]$.

7. The Cartesian Multiplier $(x + z)^m$

Fekete and Pólya† have announced without proof that if m is a sufficiently large positive integer, $(x + 1)^m$ will be what I have elsewhere‡ styled a *Cartesian Multiplier* for $f(x)$, i. e., a polynomial such that the number of variations of sign presented by the coefficients in the product of $f(x)$ by the Cartesian Multiplier is equal to the exact number of positive roots of $f(x)$. A proof similar to that of Theorem 4 would establish the following theorem analogous to the theorem of Fekete and Pólya; we shall here, however, give a proof which depends on the preceding theorems of this paper.

Theorem 9. *For each positive number* z *there exists a positive integer* k *such that for all positive integers* m *greater than* k *the polynomial* $(x + z)^m$ *is a Cartesian Multiplier for the given polynomial* $f(x)$.

There is nothing in the proofs of Theorems 1–6 which would prevent our substituting for $f(x)$ a polynomial whose coefficients are functions of z. In particular, let us form the array (4) for the polynomial $f(x - z)$. For our former proviso that $f(z) \neq 0$ we must now substitute $f(0) \neq 0$. One can, however, easily change the following statements so as to include the case $f(0) = 0$.

In the array (4) for $f(x - z)$, the elements in and beyond the principal diagonal are given by the analogue of formula (12),

$$\bar{A}_{ij} = \frac{1}{(i-1)!}\frac{d^{(i-1)}}{dx^{(i-1)}}[x^{i+j-n-1}f(x - z)]\Big|_{z=x} \qquad (i > 0,\ i + j > n). \tag{15}$$

The transformation $x - z = X$ changes this into

$$\bar{A}_{ij} = \frac{1}{(i-1)!}\frac{d^{(i-1)}}{dX^{(i-1)}}[(X + z)^{i+j-n-1}f(X)]_{X=0}. \tag{16}$$

Thus *the terms* $\bar{A}_{ij}$ *of the array* (4) *formed for* $f(x - z)$ *on a diagonal beyond the principal diagonal, i. e., terms for which* $i + j = m + n + 1$, *where* m *is a positive integer, are the coefficients of the powers of* x *in the polynomial*

$$(x + z)^m f(x).$$

* We may include the cases where either a, or b, or both are roots of $f(x)$ by making $[a, b]$ include b in all cases, but a only when $f(a) \neq 0$.

† Loc. cit., p. 120.

‡ *An Extension of Descartes' Rule of Signs*, Mathematische Annalen, vol. 73 (1912), p. 424.

By Theorem 5 applied to $f(x-z)$, the integer k can be determined so that for $m = k+1$ the number of variations of sign in the corresponding diagonal sequence is equal to the number of roots of $f(x-z)$, as a polynomial in x, in $[z, +\infty]$. By Theorem 2 the same must be true for all values of $m > k$. Our proof of Theorem 9 is now complete if we observe first that all the variations of sign occur in the diagonal parts of these sequences, and then note that there is a one-to-one correspondence between the roots of $f(x-z)$ in $[z, +\infty]$, and those of $f(x)$ in $[0, +\infty]$.

We could form an array of the coefficients of $(x+z)^m f(x)$ for $m = 0, 1, \cdots$, and sequences formed according to certain easily inferred laws would have properties corresponding to those for Laguerre Sequences given in Theorems 1–6. The reader will, however, find little difficulty in carrying this out for himself.